Path Integral methods for Stochastic Differential Equations

Carson C Chow and Michael A Buice

TNJC 11th December 2012

I Introduction

Motivation

- Study SDEs of the form

$$\frac{dx}{dt} = f(x) + g(x)\eta(t)$$

- Want to know *moments* (*e.g.* $\langle x(t) \rangle$, $\langle x(t)x(t') \rangle$) and *probability density function* (pdf, $p(x,t)$)
- Can use Langevin and Fokker-Planck equations to study, but perturbation methods can be difficult to apply

I Introduction

Outline

2. Introduce moment generating functionals ($Z[\lambda]$), distribution of functions ($P[x(t)]$)
3. Path integrals to compute moment generating functional of SDE, using Ornstein-Uhlenbeck process as example
4. Perturbation methods using Feynman diagrams
5. Derive equations for the density $p(x, t)$

II Moment generating functionals

Moment generating function

- For a single random variable X, the *moments* ($\langle X \rangle = \int x^n P(x)\, dx$) are obtained from the MGF

$$Z(\lambda) = \langle e^{\lambda x} \rangle = \int e^{\lambda x} P(x)\, dx$$

by taking derivatives

$$\langle X^n \rangle = \frac{1}{Z(0)} \frac{d^n}{d\lambda^n} Z(\lambda) \Big|_{\lambda=0}$$

- MGF contains all information about RV X, alternative to studying the pdf directly.

II Moment generating functionals

Cumulant generating function

- Define $W(\lambda) = \log Z(\lambda)$, then

$$\langle X^n \rangle_C = \frac{d^n}{d\lambda^n} W(\lambda)|_{\lambda=0}$$

  are the *cumulants* of RV X
- As with MGF, contains all information about X, and is sometimes more convenient. For example

$$\begin{aligned} \langle X \rangle_C &= \langle X \rangle \\ \langle X^2 \rangle_C &= \langle X^2 \rangle - \langle X \rangle^2 = \text{var}(x) = \text{2nd central moment} \\ \langle X^3 \rangle_C &= \langle X^3 \rangle - 3\langle X^2 \rangle \langle X \rangle + 2\langle X \rangle^3 = \text{3rd central moment} \\ &\vdots \end{aligned}$$

- Higher order cumulants are neither moments or central moments

II Moment generating functionals

Moment generating function(al)

- For an n-dimensional random variable $\mathbf{x} = (x_1, \ldots, x_n)$, the generating functional is

$$Z(\lambda) = \langle e^{\lambda \cdot \mathbf{x}} \rangle = \int \prod_{i=1}^{n} dx_i e^{\lambda \cdot \mathbf{x}} P(\mathbf{x})$$

  for $\lambda = (\lambda_1, \ldots, \lambda_n)$.
- kth order moments are obtained via

$$\left\langle \prod_{i=1}^{k} x_{(i)} \right\rangle = \frac{1}{Z(0)} \prod_{i=1}^{k} \frac{\partial^n}{\partial \lambda_{(i)}} Z(\lambda) \bigg|_{\lambda=0}$$

- As before, the cumulant generating function is $W(\lambda) = \log Z(\lambda)$

II Moment generating functionals

Moment generating functional

- Identify with each x_i in $\mathbf{x}$ a time, $t = ih$, such that $x_i = x(ih)$ and let total time $T = nh$, splitting interval $[0, T]$ into n segments of length h
- Take limit $n \to \infty$ (with $h = T/n$) such that $x_i \to x(ih) = x(t)$, $\lambda_i \to \lambda(t)$ and $P(\mathbf{x}) \to P[x(t)] = \exp(-S[x(t)])$ for some functional $S[x]$, called the *action*
- Instead of summing over all points in $\mathbb{R}^n$ $\left(\int \prod_{i=1}^{n} dx_i\right)$, sum over all curves $\left(\int \mathcal{D}x(t)\right)$, giving the MGF:

$$Z[\lambda] = \int \mathcal{D}x(t)\, e^{-S[x] + \int \lambda(t)x(t)\,dt}$$

II Moment generating functionals

Moment generating functional

- Note: inner product becomes $\mathbf{x} \cdot \mathbf{y} \to \int x(t)y(t)\,dt$
- Moments can be obtained via

$$\left\langle \prod_{i=1}^{k} x(t_{(i)}) \right\rangle = \frac{1}{Z[0]} \prod_{i=1}^{k} \frac{\delta}{\delta\lambda(t_{(i)})} Z[\lambda] \Big|_{\lambda(t)=0}$$

- Cumulant generating functional again

$$W[\lambda] = \log(Z[\lambda])$$

II Moment generating functionals

The functional derivative $\frac{\delta F[\varphi]}{\delta \varphi}$

- Extension of *directional derivative* for functions $f : \mathbb{R}^n \to \mathbb{R}$:

$$\nabla_v f(\mathbf{x}) = \nabla f \cdot \mathbf{v} = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{v})}{\epsilon}$$

  gives rate of change in direction vector **v** at point **x**.

- Can compute using a test function $f(x)$

$$\left\langle \frac{\delta F}{\delta \varphi}, f(x) \right\rangle = \frac{d}{d\epsilon} F[\varphi + \epsilon f] \bigg|_{\epsilon=0}$$

- Example:

$$W[\lambda] = \int \frac{1}{2} \lambda^2(t)\, dt; \quad \frac{\delta W}{\delta \lambda} = \lambda(t)$$

$$F[\varphi] = e^{\int \varphi(x) g(x)\, dx}; \quad \frac{\delta F}{\delta \varphi} = g(x) e^{\int \varphi(x) g(x)\, dx}$$

II Moment generating functionals

Gaussian RVs in one dimension

- RV $X \sim N(a, \sigma^2)$ has MGF

$$Z(\lambda) = \int_{-\infty}^{\infty} \exp\left[\frac{-(x-a)^2}{2\sigma^2} + \lambda x\right] dx = \sqrt{2\pi}\sigma \exp(\lambda a + \lambda^2\sigma^2/2),$$

  obtained by completing the square.
- And has cumulant GF

$$W(\lambda) = \lambda a + \frac{1}{2}\lambda^2\sigma^2 + \log(Z(0))$$

  so cumulants are $\langle x \rangle_C = a, \langle x^2 \rangle_C = \text{var}\, X = \sigma^2$ and $\langle x^k \rangle_C = 0$ for all $k > 2$.

II Moment generating functionals

Gaussian RVs in n dimensions

- The n dimensional RV $X \sim N(0, K)$, with covariance matrix K, has MGF

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_{jk} x_j K^{-1}_{jk} x_k + \sum_j \lambda_j x_j}\, dx$$

- Since K is symmetric positive definite (then so is K^{-1}) we can diagonalise in orthonormal coordinates, giving

$$Z(\lambda) = [2\pi \det(K)]^{n/2} e^{\frac{1}{2}\sum_{jk} \lambda_j K_{jk} \lambda_k}$$

Infinite dimensional

- MGF is

$$Z[\lambda] = \int \mathcal{D}x(t) e^{-\frac{1}{2}\int x(s) K^{-1}(s,t) x(t) ds dt + \int \lambda(t) x(t) dt} = Z[0] e^{\frac{1}{2}\int \lambda(s) K(s,t) \lambda(t) ds dt}$$

II Moment generating functionals

Gaussian RVs and Wick's theorem

- Relate higher order central moments of multivariate normally distributed **x** to products of second order central moments

$$\left\langle \prod_{i=1}^{k} x_{(i)} \right\rangle = \begin{cases} 0, & k \text{ odd} \\ \sum_{\sigma\in A} K_{\sigma(1)\sigma(2)} K_{\sigma(3)\sigma(4)} \cdots K_{\sigma(k-1)\sigma(k)}, & k \text{ even} \end{cases}$$

  for $A = \{\text{all pairings of } x_{(i)}\}$. For $k = 2l$ sum will contain $(2l-1)!/[2^{l-1}(l-1)!]$ terms.
- Example: for $k = 4$

$$\langle x_1 x_2 x_3 x_4 \rangle = K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}$$

II Moment generating functionals

Quantum mechanics

Sum over paths $x(t)$: $K(a,b) = \int \mathcal{D}x(t) e^{2\pi i S[x]/h}$ where $|K(a,b)|^2$ gives the probability particle with action $S = \int_{t_a}^{t_b} L(x(t), \dot{x}(t), t)\, dt$ travels from point a to b.

Quantum field theory

Sum over fields $\varphi(\mathbf{x}, t)$:

$$S[\varphi] = \int \varphi(\mathbf{t}) K^{-1}(\mathbf{t}, \mathbf{t}') \varphi(\mathbf{t}) d^d t d^d t' + g \int \varphi^4(\mathbf{t}) d^d t$$

Statistical mechanics

The sum over all states $Z = \sum_q e^{-\beta S[q]}$ is called the partition function. $Z[J]$ is the partition function of QFT

II Moment generating functionals

Quantum mechanics

When S is large compared with $h/2\pi$ the integral $\int \mathcal{D}x(t)e^{2\pi iS[x]/h}$ is a rapidly oscillating exponential – method of stationary phase says only curves for which $\frac{\delta S}{\delta x} = 0$ contribute $\Rightarrow$ principle of least action in classical mechanics

Quantum field theory

- For the case where we can describe state of system in terms of mechanical variables (e.g. atoms in a periodic latte, crystal), each point is described by a quantum harmonic oscillator. If sensible to take a continuum approximation $\rightarrow$ quantum field theory. Higher modes, excited states, in the coupled oscillators have particle like behaviour.
- In other cases, each point in space is described other variables, e.g. electromagnetism, but may still be quantised as oscillators. Excited states of these fields are called *bosons*, such as the photon.

III Application to SDEs

Repeat for an SDE

- Construct generating functional for SDEs of the form

$$\frac{dx}{dt} = f(x,t) + g(x)\eta(t) + y\delta(t-t_0),$$

  for $t \in [0, T]$.
- Discretize in time steps h (Ito interpretation)

$$x_{i+1} - x_i = f_i(x_i)h + g_i(x_i)w_i\sqrt{h} + y\delta_{i,0}$$

- Each w_i is Guassian with $\langle w_i \rangle = 0$ and $\langle w_i w_j \rangle = \delta_{ij}$

III Application to SDEs

Probability generating functional

- PDF given the random walk w_i:

$$P[x|w;y] = \prod_{i=0}^{n} \delta[x_{i+1} - x_i + f_i(x_i)h - g_i(x_i)w_i\sqrt{h} - y\delta_{i,0}]$$

- Take Fourier transform

$$P[x|w;y] = \int \prod_{j=0}^{N} \frac{dk_j}{2\pi} e^{-i\sum_j k_j(x_{j+1}-x_j-f_j(x_j)h-g_j(x_j)w_j\sqrt{h}-y\delta_{j,0})}$$

- Using the law of total probability and completing the square:

$$P[x|y] = \int \prod_{j=0}^{N} \frac{dk_j}{2\pi} e^{-\sum_j(ik_j)\left(\frac{x_{j+1}-x_j}{h}-f_j(x_j)-y\delta_{j,0}/h\right)h+\sum_j \frac{1}{2}g_j^2(x_j)(ik_j)^2 h}$$

III Application to SDEs

Continuum limit

- Again let $h \to 0$ with $N = T/h$, replace ik_i with $\tilde{x}(t)$ and $\dfrac{x_{j+1} - x_j}{h}$ with $\dot{x}(t)$:

$$P[x(t)|y, t_0] = \int \mathcal{D}\tilde{x}(t) e^{-\int [\tilde{x}(t)(\dot{x}(t) - f(x(t),t) - y\delta(t-t_0)) - \frac{1}{2}\tilde{x}^2 g^2(x(t),t)] dt}$$

- $\tilde{x}(t)$ represents wave number k, proportional to momentum p, thus can write down a moment generating functional for position and momumentum space:

$$Z[J, \tilde{J}] = \int \mathcal{D}x(t)\mathcal{D}\dot{x}(t) e^{-S[x,\tilde{x}] + \int \tilde{J}x\, dt + \int J\tilde{x}\, dt}$$

- Typo in set of equations below (6)?

III Application to SDEs

More generally...

- Instead of $g(x)\eta(t)$, with $\eta(t)$ white noise, an SDE having noise process with cumulant $W[\lambda(t)]$ will have PDF:

$$P[x(t)|y,t_0] = \int \mathcal{D}\eta(t)\delta[\dot{x}(t) - f(x,t) - \eta(t) - y\delta(t-t_0)]e^{-S[\eta(t)]}$$
$$= \int \mathcal{D}\eta(t)\mathcal{D}\tilde{x}(t)e^{0\int \tilde{x}(t)(\dot{x}(t)-f(x,t)-y\delta(t-t_o))\,dt+W[\tilde{x}(t)]}$$

- If $\eta(t)$ is delta correlated ($\langle\eta(t)\eta(t')\rangle = \delta(t-t')$) then $W[\tilde{x}(t)]$ can be Taylor expanded in both $x(t)$ and $\tilde{x}(t)$:

$$W[\tilde{x}(t)] = \sum_{n=1,m=0}^{\infty} \frac{v_{nm}}{n!}\int \tilde{x}^n(t)x^m(t)\,dt$$

– summation over n starts at one because $W[0] = \log(Z[0]) = 0$?
– delta correlated means no mixed derivative terms in finite dimensional equivalent?

A. Ornstein-Uhlenbeck Process

For example...

- The OU process has the action

$$S[x,\tilde{x}] = \int \left[\tilde{x}(t)(\dot{x}(t) + ax(t) - y\delta(t - t_0)) - \frac{D}{2}\tilde{x}^2(t) \right] dt$$

- *G* and OU's moments could found immedately, since action is quadratic, demonstrate perturbation method to motivate the method more generally
- Break action into 'free' and 'interacting' terms. Free terms would represent a particle without any interaction with a field or potential, and would have a quadratic action

A. Ornstein-Uhlenbeck Process

Green's functions

- G, the linear response function, or propagator, is a Green's function:

$$\left(\frac{d}{dt} + a\right) G(t, t') = \delta(t - t')$$

- $G(t, t')$ is equivalent to $K(t, t')$ from the generic Gaussian stochastic process in Section II (equation 2), also called the correlator, and in QM would represent probability a particle travelling from one point to another
- The free generating functional is

$$Z_F[J, \tilde{J}] = \int \mathcal{D}x(t)\mathcal{D}\tilde{x}(t)e^{-\int dtdt'\tilde{x}(t)G^{-1}(t,t')x(t)\,dt + \int \tilde{x}(t)J(t)\,dt + \int x(t)\tilde{J}(t)\,dt}$$

so, from (2):

$$Z_F[J, \tilde{J}] = e^{\int \tilde{J}G(t,t')J\,dtdt'}$$

A. Ornstein-Uhlenbeck Process

Green's functions

- Solve for G:

$$G(t,t') = H(t-t')e^{-a(t-t_0)}$$

- The free moments are

$$\left\langle \prod_{ij} x(t_i)\tilde{x}(t_j) \right\rangle_F = \prod_{ij} \frac{\delta}{\delta \tilde{J}(t_i)} \frac{\delta}{\delta J(t_j)} e^{\int \tilde{J}(t)G(t,t')J(t')\,dtdt'} \Bigg|_{J=\tilde{J}=0}$$

- Note:

$$\langle x(t_1)\tilde{x}(t_2)\rangle_F = \frac{\delta}{\delta \tilde{J}(t_1)} \frac{\delta}{\delta J(t_2)} e^{\int \tilde{J}(t)G(t,t')J(t')\,dtdt'} \Bigg|_{J=\tilde{J}=0} = G(t_1,t_2)$$

and $\langle \tilde{x}(t_1)\tilde{x}(t_2)\rangle_F = \langle x(t_1)x(t_2)\rangle_F = 0$, so Wick's theorem means all higher order free moments must have equal numbers of x's as $\tilde{x}$'s.

A. Ornstein-Uhlenbeck Process

Perturbed generating functional

- Equation (7) can also be written

$$Z[J,\tilde{J}] = Z_F[0,0] + \sum_{m=1}^{\infty} \frac{1}{m!}\langle \mu^m \rangle_F$$

  so we can now evaluate $Z[J,\tilde{J}]$ in terms of the free moments.
- (9) and the equation below take some work

Results

- Once the MGF is determined so is the cumulant generating functional

$$W[J,\tilde{j}] = y\int \tilde{J}(t)G(t,t_0)\,dt_1 + \int \tilde{J}(t')J(t'')G(t',t'')\,dt'dt'' + \frac{D}{2}\int \tilde{J}(t')\tilde{J}(t'')G($$

- The moments/cumulants can be read immediately from $W[J,\tilde{J}]$ in terms of the propagator
- Knowing $G(t,t')$ allows the moments to be computed explicitly

IV Perturbation methods and Feynman Diagrams

The MGF expanded about the free action

- As in the OU process, split action into linear and non-linear parts $S = S_F + S_I$:

$$Z[J, \tilde{J}] = \int \mathcal{D}x(t)\mathcal{D}\tilde{x}(t)e^{-S_F - S_I + \int \tilde{J}x + \int J\tilde{x}}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mu^n \rangle_F$$

  with $\mu = S_I + \int \tilde{J}x\, dt + \int J\tilde{x}\, dt$
- Expand S_I:

$$S_I = \sum_{n\geq 2, m\geq 0} V_{mn} = \sum_{n\geq 2, m\geq 0} \int x^m \tilde{x}^n\, dt$$

- Remember to distinguish between free moments given by Z_F (to which Wick's theorem applies) and those given by Z (to which Wick's theorem does not apply).

And a diagrammatic equivalent

- With each V_{mn} in S_I associate an *internal vertex* having m entering edges and n exiting edges; these elements are connected with one another in *all possible ways* (multinomial theorem) in the expansion above
- $\int J\tilde{x}$ and $\int \tilde{J}x$ terms contribute respectively entering and exiting *external vertices*
- Edges connecting vertices correspond to a pairing between an $x(t)$ and $\tilde{x}(t)$
- *e.g.* the OU process...

$$S_I = \int dt\, y\delta(t - t_0)\tilde{x}(t) + \int dt\, \frac{D}{2}\tilde{x}^2(t)$$

Wick's theorem

- *All possible ways:* only free moments with equal numbers of x and $\tilde{x}$'s are non-zero – those of the form $\langle \prod_{i=1}^{k} x(t_i)\tilde{x}(t_i')\rangle_F$
- For example, the coupling between external vertex $\int \tilde{J}x\,dt$ and internal vertex $\int \delta(t-t_0)y\tilde{x}(t)\,dt$ in Z contributes:

$$Z = \left\langle \int dtdt'\,\tilde{J}(t)x(t)y\delta(t'-t_0)\tilde{x}(t')\right\rangle_F + \text{all other terms}$$

$$= \int dtdt'\,\tilde{J}(t)y\delta(t'-t_0)\,\langle x(t)\tilde{x}(t')\rangle_F + \text{all other terms}$$

$$= \int dt\,y\tilde{J}(t)G(t,t_0) + \text{all other terms}$$

- Vertices in diagram are assigned temporal index t_k

Computing moments with Feynman diagrams

- Recall

$$\left\langle \prod_{i=1}^{N} \prod_{j=1}^{M} x(t_i)\tilde{x}(t_j) \right\rangle = \frac{1}{Z[0,0]} \frac{\delta}{\delta J(t_i)} \frac{\delta}{\delta \tilde{J}(t_j)} Z \bigg|_{J=\tilde{J}=0}$$

  $\Rightarrow$ only terms/diagrams in expansion for Z having N entering and M exiting external vertices will contribute to that moment
  $\Rightarrow$ moments can be computed by writing down all possible diagrams with requiste number of external vertices
- In OU only a finite number of diagrams need be considered and the exact mean and covariance can be determined immediately

Computing moments with Feynman diagrams

- For the process $\dot{x} = -ax + bx^2 + y\delta(t - t_0) + \sqrt{D}x^{n/2}\eta(t)$:

$$S_I = -y\tilde{x}(t_0) - b\int dt\,\tilde{x}(t)x^2(t) - \int \tilde{x}^2 x^n \frac{D}{2}$$

  the components and example diagrams are Figures 1 and 2

- The mean and covariance are

$$\begin{aligned}\langle x(t)\rangle =& yG(t,t_0) + bD\int G(t,t_1)G(t_1,t_2)^2\,dt_1dt_2\\ &+ by^2\int G(t,t_1)G(t_1,t_0)^2\,dt_1 + \dots\\ \langle x(s)x(t)\rangle =& D\int G(s,t_1)G(t,t_1)\,dt_1 + y^2G(s,t_0)G(t,t_0)\\ &+ 2bDy\int G(s,t_1)G(t,t_2)G(t_1,t_2)G(t_1,t_0)\,dt_1dt_2 + \dots\end{aligned}$$

IV Perturbation methods and Feynman Diagrams

Which terms contribute the most?

- If some terms in S_I ($v_{mn} \int x^n \tilde{x}^m, m \geq 2$) are small, let each such vertex contribute a small parameter α
- Perform expansion in orders of α – 'weak coupling expansion'
- *e.g.* in QED coupling is related to change of electron (e):

$$\alpha \approx 1/137 = \text{fine structure constant}$$

IV Perturbation methods and Feynman Diagrams

Small noise expansion

- Scale entire exponent in MGF by some factor h

$$Z = \int \mathcal{D}x(t)\mathcal{D}\tilde{x}(t)e^{-\frac{1}{h}(S-\int \tilde{J}x-\int J\tilde{x})}$$

- Each vertex of S_I gains a factor of $1/h$ and each edge of S_F gains a factor $h \Rightarrow$ can expand in powers of h
- Can show $h^{E-I+1} = h^{E-L+1} \Rightarrow$ expand in number of loops in diagrams
- Deterministic equation has no loops – all diagrams are trees: 'classical edges'
- $\Rightarrow$ construct moments with same vertices and diagrams as in Figure 1 and 2 but replace edges with classical ones
- $\Rightarrow$ a 'semi-classical' expansion

V Connection to Fokker-Planck Equation

Computing density $p(x, t)$

- Let $U(x_1, t_1|x_0, t_0)$ be the transition probability, then

$$\begin{aligned} U(x_1, t_1|x_0, t_0) &= \int \mathcal{D}x(t)\delta(x(t_1) - x_1)P[x(t)] \\ &= \frac{1}{2\pi i}\int d\lambda \int \mathcal{D}x(t)e^{-\lambda(x(t_1)-x_1)}P[x(t)] \\ &= \frac{1}{2\pi i}\int d\lambda \int \mathcal{D}x(t)e^{-\lambda(x_1-x_0)}e^{\lambda(x(t_1)-x_0)}P[x(t)] \\ &= \frac{1}{2\pi i}\int d\lambda \int \mathcal{D}x(t)e^{-\lambda(x_1-x_0)}Z_{CM}(\lambda) \end{aligned}$$

- Z_{CM} gives moments of $x(t_1) - x_0$ given $x(t_0) = x_0$
- Initial condition is incorporated in $P[x(t)]$ as done previously – means $P[x(t)]$ may be given by a path integral over $\tilde{x}(t)$.

Computing density $p(x, t)$

- Using:

$$Z_{CM}(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \langle (x(t_1) - x_0)^n \rangle_{x(t_0)=x_0}$$

$$\frac{1}{2\pi i} \int d\lambda \, e^{-\lambda(x_1 - x_0)} \lambda^n = \left(-\frac{\partial}{\partial x_1} \right)^n \delta(x_1 - x_0)$$

  U becomes

$$U(x_1, t_1 | x_0, t_0) = \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x_1} \right)^n \langle (x(t_1) - x_0)^n \rangle_{x(t_0)=x_0} \right) \delta(x_1 - x_0)$$

Computing density $p(x,t)$

- Can derive a relation for $p(x,t)$:

$$p(y,t+\Delta t) = \int U(x,t+\Delta t|y',t)p(y',t)\,dy'$$
$$= \int \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial y}\right)^n \langle (x(t_1) - y')^n \rangle_{x(t)=y'}\right) \delta(y-y')p(y',t)\,dy'$$
$$= \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial y}\right)^n \langle (x(t_1) - y)^n \rangle_{x(t)=y}\right) p(y,t)$$

V Connection to Fokker-Planck Equation

Computing density $p(x, t)$

- Can derive a PDE for $p(x, t)$:

$$\frac{\partial p(y,t)}{\partial t}\Delta t = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial y}\right)^n \langle (x(t_1) - y)^n \rangle_{x(t)=y} p(y,t) + O(\Delta t^2)$$

$$\Rightarrow \frac{\partial p(y,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial y}\right)^n D_n(y,t) p(y,t)$$

  as $\Delta t \to 0$. The Kramers-Moyal expansion

- D_n are

$$D_n(y,t) = \lim_{\Delta t \to 0} \frac{\langle (x(t+\Delta t) - y)^n \rangle}{\Delta t}\bigg|_{x(t)=y}$$

- D_n are computed from the SDE

V Connection to Fokker-Planck Equation

Computing density $p(x, t)$

- Example: for the Ito process

$$dx = f(x, t)dt + g(x, t)dB_t$$

  we can compute $D_1(y, t) = f(y, t)$ and $D_2(y, t) = g(y, t)^2$, $D_n = 0$ for $n > 2$.

- Hence the PDE becomes a Fokker-Planck equation

$$\frac{\partial p(y, t)}{\partial t} = \left(\frac{\partial}{\partial y} D_1(y, t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} D_2(y, t) \right) p(y, t)$$

V Connection to Fokker-Planck Equation

Computing density $p(x, t)$

- Compute $p(x, t) = U(x, t|0, 0)$ as

$$p(x,t) = \frac{1}{2\pi i} \int d\lambda \, e^{-\lambda x} Z_{CM}(\lambda)$$
$$= \frac{1}{2\pi i} \int d\lambda \, e^{-\lambda x} \exp\left[\sum_{n=1} \frac{1}{n!} \lambda^n \langle x(t)^n \rangle_C \right]$$

- For OU we know the cumulants hence

$$p(x,t) = \sqrt{\frac{a}{\pi D(1 - e^{-2a(t-t_0)})}} \exp\left(\frac{-a(x - ye^{-a(t-t_0)})^2}{D(1 - e^{-2a(t-t_0)})} \right)$$

VI Conclusion

One extra reference

1 R. Feynman, A. Hibbs, *Quantum Mechanics and Path Integrals*. Dover, emended edition, 2005.
Provides physical context. Final chapter discusses similar material to this paper